PMM U.S.S.R., Vol. 46, No. 5, pp. 704-706, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 62-50

ON FUNCTIONAL EQUATIONS IN GAMES OF ENCOUNTER AT A PRESCRIBED INSTANT*

S.V. CHISTIAKOV

Functional operators, simpler than in /1—3/, are determined in connection with games of encounter at a prescribed instant. These operators or their analogs can be used to obtain many of the fundamental results in /1—3/, in particular, iteration methods can be constructed, converging to the game's value function. As one more illustration of the capabilities of the method developed in /1—3/, the operators determined below are used to give a new proof of the well-known result /4/ on the identification of the game's value function by means of the so-called main equation.

Let the dynamics in a game of encounter at a prescribed instant be described by the system

$$x^{\cdot} = f(t, x, u, v);\ x \in R^n \tag{1}$$
$$u \in P \in \mathrm{Comp}\, R^k,\quad v \in Q \in \mathrm{Comp}\, R^m$$

Concerning the vector-valued function $f(\cdot)$ we assume the fulfillment of the following conditions: 1) $f(\cdot)$ is continuous on $(-\infty, T] \times R^n \times P \times Q$ and satisfies a local Lipschitz condition in x; 2) there exists $\lambda > 0$ such that $\|f(t, x, u, v)\| \leqslant \lambda(1 + \|x\|)$ for all $t \in (-\infty, T]$, $x \in R^n$, $u \in P$, $v \in Q$; 3) the equality

$$\max_{v\in Q} \min_{u\in P} \langle l, f(t, x, u, v)\rangle = \min_{u\in P} \max_{v\in Q} \langle l, f(t, x, u, v)\rangle$$

is valid for any $l \in R^n$, $t \in (-\infty, T]$ and $x \in R^n$. We define the operators

$$\Phi_-, \Phi_+ : C((-\infty, T] \times R^n) \to C((-\infty, T] \times R^n)$$

For any function $w(\cdot)$ continuous on $(-\infty, T] \times R^n$ and for any $t^\circ \in (-\infty, T]$, $x^\circ \in R^n$

$$\Phi_- \circ w(t^\circ, x^\circ) = \max_{t\in[t^\circ, T]} \max_{v\in Q} \inf_{u(\cdot)} w(t, x(t, t^\circ, x^\circ, u(\cdot), v))$$
$$\Phi_+ \circ w(t^\circ, x^\circ) = \min_{t\in[t^\circ, T]} \min_{u\in P} \sup_{v(\cdot)} w(t, x(t, t^\circ, x^\circ, u, v(\cdot)))$$

where the operation inf (respectively, sup) ranges over all piecewise-constant functions $u : [t^\circ, T] \to P$ ($v : [t^\circ, T] \to Q$), while the function $x(\cdot, t^\circ, x^\circ, u(\cdot), v)$ ($x(\cdot, t^\circ, x^\circ, u, v(\cdot))$) is the solution of Eq.(1) on interval $[t^\circ, T]$, with the initial condition $x(t^\circ) = x^\circ$, under the piecewise-constant control $u(\cdot)$ ($v(\cdot)$) and under the constant control v (u) on the interval $[t^\circ, T]$. As was done in /1,2/, it can be shown that the definitions of operators Φ_- and Φ_+ are well posed, and, in particular, they indeed map $C((-\infty, T] \times R^n)$ into itself. In addition, as in /1,2/, the following lemma can be established.

Lemma. The inequalities $\Phi_- \circ w(\cdot) \geqslant w(\cdot)$ and $\Phi_+ \circ w(\cdot) \leqslant w(\cdot)$ are valid for any function $w(\cdot) \in C((-\infty, T] \times R^n)$.

Theorem 1. Let the function $w^\circ(\cdot) \in C((-\infty, T] \times R^n)$ and be continuously differentiable in the domain $(-\infty, T) \times R^n$. Then assertions 1° and 2° are equipotent:

1°. In domain $(-\infty, T) \times R^n$ the function $w^\circ(\cdot)$ satisfies the equation

$$\max_{v\in Q} \min_{u\in P} \left\langle \frac{\partial w}{\partial x}(t, x), f(t, x, u, v)\right\rangle + \frac{\partial w}{\partial t}(t, x) = 0 \tag{2}$$

2°. The function $w^\circ(\cdot)$ is the common fixed point of operators Φ_- and Φ_+.

Proof. Let us show that if assertion 1° is valid, then, for example, $\Phi_+ \circ w^\circ(\cdot) = w^\circ(\cdot)$. To do this, with due regard to the Lemma, it suffices to show that $\Phi_+ \circ w^\circ(\cdot) \geqslant w^\circ(\cdot)$. The latter inequality is automatically fulfilled if

---

*Prikl.Matem.Mekhan.,46,No.5,874-877,1982

$$\sup_{v(\cdot)} w^\circ(t, x(t, t^\circ, x^\circ, u, v(\cdot))) \geqslant w^\circ(t^\circ, x^\circ) \tag{3}$$

for any $t^\circ \in (-\infty, T]$, $x^\circ \in R^n$, $u \in P$ and $t \in [t^\circ, T]$. Inequality (3) is trivial when $t = t^\circ$; therefore, let $t^\circ < t \leqslant T$. Since function $w^\circ(\cdot)$ is of class C^1, (3) is equivalent to the inequality

$$\sup_{v(\cdot)} \int_{t^\circ}^{t} h(\tau, v(\cdot))\, d\tau \geqslant 0; \quad h(\tau, v(\cdot)) = \left\langle \frac{\partial w^\circ}{\partial x}(\tau, x(\tau, v(\cdot))), f(\tau, x(\tau, v(\cdot)), u, v(\tau)) \right\rangle + \tag{4}$$
$$\frac{\partial w^\circ}{\partial t}(\tau, x(\tau, v(\cdot))), \; x(\tau, v(\cdot)) = x(\tau, t^\circ, x^\circ, u, v(\cdot))$$

(here the arbitrary $t^\circ \in (-\infty, T)$, $x^\circ \in R^n$, $u \in P$ and $t \in (t^\circ, T])$ are reconed fixed). By assumption, function $w^\circ(\cdot)$ satisfies Eq. (2) in the domain $(-\infty, T) \times R^n$; therefore, for any $\delta = \delta(k) = (t - t^\circ)/k$ we can find a piecewise-constant control $v_\delta(\cdot)$ ($v_\delta(\tau) = v_i$ for $\tau \in [t^\circ + (i-1)\delta, t^\circ + i\delta)$, $i = 1, 2, \ldots, k-1$ and $v_\delta(\tau) = v_k$ for $\tau \in [t - \delta, t]$) such that

$$h(t^\circ + (i-1)\delta, v_\delta(\cdot)) \geqslant 0, \; i = 1, 2, \ldots, k \tag{5}$$

In view of the continuity of the functions $\partial w^\circ(\cdot)/\partial x$, $\partial w^\circ(\cdot)/\partial t$ and $f(\cdot)$ in their domains, as well as in view of the uniform boundedness and equicontinuity of the set of solutions $x(\tau, t^\circ, x^\circ, u, v(\cdot))$, $\tau \in [t^\circ, t]$, of system (1), corresponding to all possible piecewise-constant controls $v: [t^\circ, t] \to Q$, it follows from (5) that for any $\varepsilon > 0$ we can find such $\delta = \delta(k)$ and the corresponding control $v_\delta(\cdot)$ which will ensure the fulfillment of the inequality

$$h(\tau, v_\delta(\cdot)) \geqslant -\varepsilon/(t - t^\circ), \; \forall \tau \in [t^\circ, t]$$

Hence, with due regard to the definition of function $h(\tau, v_\delta(\cdot))$, $\tau \in [t^\circ, t]$, and to the arbitrariness of $\varepsilon > 0$, we have (4). Thus, from the validity of assertion 1° it follows that $\Phi_+ \circ w^\circ(\cdot) = w^\circ(\cdot)$. Analogously, with due regard to assumption 3) on $f(\cdot)$, it can be established that from the validity of assertion 1° follows $\Phi_- \circ w^\circ(\cdot) = w^\circ(\cdot)$. We take the implication $1^\circ \Rightarrow 2^\circ$ as proved.

Let us now prove that $2^\circ \Rightarrow 1^\circ$. We assume the contrary. For example, let assertion 2° be valid, but let there exist $t^\circ \in (-\infty, T)$ and $x^\circ \in R^n$ such that the expression on the left-hand side of (2) is greater than zero. Then, because the functions $\partial w^\circ(\cdot)/\partial x$, $\partial w^\circ(\cdot)/\partial t$ and $f(\cdot)$ are continuous, we can find neighborhoods $S(t^\circ) \subset (-\infty, T)$ and $S(x^\circ) \subset R^n$ of points t° and x°, as well as a control $v^\circ \in Q$, such that for any $u \in P$, $t \in S(t^\circ)$ and $x \in S(x^\circ)$

$$\left\langle \frac{\partial w^\circ}{\partial x}(t, x), f(t, x, u, v^\circ) \right\rangle + \frac{\partial w^\circ}{\partial t}(t, x) \geqslant \alpha > 0$$

Since the set of solutions $x(\tau, t^\circ, x^\circ, u(\cdot), v^\circ)$, $\tau \in [t^\circ, T]$, of system (1), corresponding to all possible piecewise-constant controls $u: [t^\circ, T] \to P$, is equicontinuous, we can find $\vartheta \in S(t^\circ)$ $(t^\circ < \vartheta \leqslant T)$ such that

$$\inf_{u(\cdot)} \int_{t^\circ}^{\vartheta} h(\tau, u(\cdot))\, d\tau \geqslant \alpha > 0; \quad h(\tau, u(\cdot)) =$$
$$\left\langle \frac{\partial w^\circ}{\partial x}(\tau, x(\tau, u(\cdot))), f(\tau, x(\tau, u(\cdot)), u(\tau), v^\circ) \right\rangle +$$
$$\frac{\partial w^\circ}{\partial t}(\tau, x(\tau, u(\cdot))), \; x(\tau, u(\cdot)) = x(\tau, t^\circ, x^\circ, u(\cdot), v^\circ)$$

From this inequality, in its own turn, follows the inequality $\Phi_- \circ w^\circ(t^\circ, x^\circ) > w^\circ(t^\circ, x^\circ)$ which contradicts the fact that $w^\circ(\cdot)$ is a fixed point of operator Φ_-. Analogously, assuming that assertion 2° is valid and that the expression on the left-hand side of (2) is less than zero, we arrive at a contradiction with the fact that $w^\circ(\cdot)$ is a fixed point of operator Φ_+. The theorem has been proved.

We now assume that in the game of encounter at a prescribed instant, described by system (1), the gain of the maximizing player, who has the choice of $v \in Q$ at his disposal, and, respectively, the loss of the minimizing player, who has the choice of $u \in P$ at his disposal, are determined by the quantity

$$H(x(T)), \; H(\cdot) \in C(R^n) \tag{6}$$

Then, the next theorem can be established by analogy with /3/.

Theorem 2. Let the function $u^\circ(\cdot) \in C((-\infty, T] \times R^n)$. Then the assertions 1° and 2° are equipotent:

1°. Function $w^{\circ}(\cdot)$ is the value function of game (1), (6).

2°. Function $w^{\circ}(\cdot)$ satisfies the boundary condition

$$w^{\circ}(T, x) = H(x) \tag{7}$$

and is a common fixed point of operators Φ_- and Φ_+.

From Theorems 1 and 2 follows the well-known result /4/ on the identification of the value function by means of the main equation (2).

Theorem 3. Let the function $w^{\circ}(\cdot) \in C((-\infty, T] \times R^n)$ and be continuously differentiable in the domain $(-\infty, T) \times R^n$. Then the following assertions 1° and 2° are equipotent:

1°. Function $w^{\circ}(\cdot)$ is the value function of game (1), (6).

2°. Function $w^{\circ}(\cdot)$ satisfies Eq.(2) in the domain $(-\infty, T) \times R^n$ and satisfies boundary condition (7).

REFERENCES

1. CHENTSOV A.G., On a game problem of encounter at a prescribed instant. Mat. Sb., Vol.99, No.3, 1976.
2. CHISTIAKOV S.V. and PETROSIAN L.A., On one approach to solving a pursuit games. Vestn. Leningrad. Gos. Univ., No.1, 1977.
3. CHISTIAKOV S.V., On solving pursuit game problems. PMM Vol.41, No.5, 1977.
4. ISAACS R., Differential Games. New York, J. Wiley and Sons. Inc., 1965.

Translated by N.H.C.